NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA4211

FACULTY OF APPLIED SCIENCE

DEPARTMENT OF APPLIED MATHEMATICS

SMA4211: FUNCTIONAL ANALYSIS

MARCH 2025 EXAMINATION

Time : 3 hours

Candidates should attempt all questions from Section A [**40 MARKS**] and ANY THREE questions in Section B [**60 MARKS**].

SECTION A

A1. Define the following;

(a) A hilbert space. [2]

(b) A banach space. [2]

(c) An open set. [2]

A2. Prove that if d is a metric on a non-empty set X, then d_1 given by

$$d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

where $x, y \in X$, is also a metric on X. [7]

A3. (a) Define an open ball. [2]

(b) Prove that arbitrary unions of open sets are open. [5]

A4. Consider the vector space $V = C[0,1]$, the set of all continuous complex-valued functions on the interval $[0,1]$. Define the inner product on V as:

$$\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}\,dt,$$

where $\overline{g(t)}$ denotes the complex conjugate of $g(t)$. Given the functions $f(t) = t$ and $g(t) = t + t^2 i$ in V.

(a) Evaluate the inner product $\langle f, g\rangle$. [4]

(b) Find the norms $\|f\|$ and $\|g\|$, where the norm is induced by the inner product, i.e.,

$$\|f\| = \sqrt{\langle f, f\rangle} \quad \text{and} \quad \|g\| = \sqrt{\langle g, g\rangle}.$$

[8]

(c) Verify that the Cauchy-Schwarz inequality holds for f and g. Specifically, show that:

$$|\langle f, g\rangle| \leq \|f\| \cdot \|g\|.$$

[3]

A5. Let $p > 1$ and $q > 1$ be real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. For any two positive real numbers a and b, prove the following inequality:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

[5]

SECTION B

Answer any **THREE** questions from this section **[60 MARKS]**

B6. (a) Define a fixed point of a function. [2]

(b) Let (X, d) be a complete metric space. Suppose $T : X \to X$ satisfies

$$d(T(x), T(y)) \leq \alpha d(x, y)$$

for every $x, y \in X$, where $0 \leq \alpha < 1$. Prove that T has a unique fixed point in X, i.e., there exists only one $x \in X$ such that $T(x) = x$.

[10]

(c) Show that $T : \mathbb{R} \to \mathbb{R}$ defined by

$$T(x) = \frac{\pi}{2} + x - \tan^{-1} x$$

has no fixed point, and

$$|T(x) - T(y)| < |x - y| \quad \text{for all } x \neq y \in \mathbb{R}.$$

[6]

(d) Find the fixed points of $f(x)$ given that

$$f(x) = x^2 + 7x + 5$$

[2]

B7. (a) Show that the space $(C[0,1], \|\cdot\|_\infty)$, where $\|x\|\infty = \sup_{0\leq t\leq 1} |x(t)|$, is not an inner product space, hence not a Hilbert space. [6]

(b) Prove that the inner product in a complex normed space X can be recovered from the polarization identity for all $\mathbf{v}, \mathbf{w} \in X$:

$$\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{4}\left(\|\mathbf{v} + \mathbf{w}\|^2 - \|\mathbf{v} - \mathbf{w}\|^2 + i\left(\|\mathbf{v} + i\mathbf{w}\|^2 - \|\mathbf{v} - i\mathbf{w}\|^2\right)\right).$$

[10]

(c) Prove that the inner product in a real normed space X can be recovered from the polarization identity for all $\mathbf{v}, \mathbf{w} \in X$:

$$\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{4}\left(\|\mathbf{v} + \mathbf{w}\|^2 - \|\mathbf{v} - \mathbf{w}\|^2\right).$$

[4]

B8. (a) Define a bounded linear operator. [2]

(b) Define $T : C[0,1] \to C[0,1]$ by

$$(Tx)(t) = t \int_0^t x(s)ds.$$

(a) Prove that T is a bounded linear operator and compute $\|T\|$. [10]

(c) Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed linear spaces and let T be a linear operator from X to Y. Show that the following statements are equivalent:

(i) T is a bounded linear operator.

(ii) T is continuous on X.

(iii) T is continuous at $0 \in X$.

[8]

B9. (a) Define a σ-algebra and a measure. [4]

(b) Show that the function $f(x) = \frac{1}{x} \cdot \chi_{(0,1]}(x)$ is not Lebesgue integrable on $(0,1]$. [3]

(c) State the Dominated Convergence Theorem and use it to evaluate

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{n \sin(x/n)}{x\,(x^2+1)} dx.$$

[9]

(d) Consider the sequence of functions $f_n(x)$ defined as:

$$f_n(x) = \begin{cases} \frac{1}{n} & x \in [0, n] \\ 0 & \text{otherwise} \end{cases}$$

(i) Show that $\int f_n(x)\,dx = 1$ for all n. [1]

(ii) Determine the pointwise limit $\lim_{n\to\infty} f_n(x)$ for all x. [1]

(iii) Explain why the behavior of $f_n(x)$ does not contradict Lebesgue's Dominated Monotone Convergence Theorem. [2]

END OF QUESTION PAPER