NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA4211

FACULTY OF APPLIED SCIENCE

DEPARTMENT OF APPLIED MATHEMATICS

SMA4211 FUNCTIONAL ANALYSIS

AUGUST 2024 SPECIAL EXAMINATION

Time : 3 hours

This paper contains **TWO** sections. Attempt **ALL** questions from Section A and any **THREE** questions from Section B.

SECTION A

Attempt **ALL** questions from this section. **[40 MARKS]**

A1. (a) Let the metrics d_1 and d_2 be defined on $C[0,1]$ as follows

$$d_1(f,g) = \max_{t \in [0,1]} |f(t) - g(t)|,$$

$$d_2(f,g) = \int_0^1 |f(t) - g(t)| \, dt,$$

where $f, g \in C[0,1]$.

Given that $f(t) = 3t^2 + 4t - 6$ and $g(t) = 3t^2 - 6t - 2$, evaluate

    (i) $d_1(f,g)$,   [4]

    (ii) $d_2(f,g)$.   [4]

(b) Evaluate the metric distance between x and y in the l^3 space where

$$x = (0, -1, 2, 0, 0, 0, 0, 0, \ldots),$$
$$y = (-2, 0, -1, 3, 2, 0, 0, 0, \ldots).$$

[5]

A2. (a) Show that in a metric space every convergent sequence is a Cauchy sequence. [4]

(b) Give a counterexample to show that the converse is not true i.e. find an example of a Cauchy sequence that is not convergent. [3]

A3. Let X be a normed space. Prove the *reverse triangle inequality*.

$$\Big|\|x\| - \|y\|\Big| \leq \|x - y\| \quad \forall\, x, y \in X.$$

[6]

A4. Let X be an inner product space. Prove the parallelogram law.

$$\|x + y\| + \|x - y\| = 2\|x\| + 2\|y\| \quad \forall x, y \in X.$$

[8]

A5. Let $C^1[0,1]$ denote the vector space of all real-valued functions defined on $[0,1]$ having first-order continuous derivatives. Show that

$$\|f\| = \sup_{0 \leq t \leq 1} \Big(|f(t)| + |f'(t)|\Big),$$

defines a norm on $C^1[0,1]$. [6]

SECTION B

Attempt any **THREE** questions from this section. **[60 MARKS]**

B6. Consider the space l^p $(1 \leq p < \infty)$, the set of all sequences $x = (\xi_i)$ such that

$$\sum_{i=1}^{\infty} |\xi_i|^p < \infty.$$

Define a metric on l^p as

$$d(x, y) = \left(\sum_{i=1}^{\infty} |\xi_i - \eta_i|^p\right)^{\frac{1}{p}}, \tag{1}$$

where $x = (\xi_i)$ and $y = (\eta_i)$.

(a) Prove the properties **M1, M2** and **M3** for the metric in (1). [5]

(b) Prove the following inequalities

    (i) Young's inequality,

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q},$$

    where α and β are positive numbers and $\frac{1}{p} + \frac{1}{q} + 1$. [4]

    (ii) Hölder's inequality,

$$\sum_{i=1}^{\infty} |\xi_i \eta_i| \leq \left(\sum_{j=1}^{\infty} |\xi_j|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{\infty} |\eta_k|^q\right)^{\frac{1}{q}}.$$

[4]

    (iii) Minkowski's inequality,

$$\left(\sum_{i=1}^{\infty} |\xi_i + \eta_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{j=1}^{\infty} |\xi_j|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty} |\eta_k|^p\right)^{\frac{1}{p}}.$$

[4]

(c) Hence, prove the property **M4** (Triangle inequality) for metric (1) to show that l^p is a metric space. [3]

B7. (a) Let $z = x + iy \in \mathbb{C}$ and let $T : \mathbb{C} \to \mathbb{C}$ be an operator given by

$$T(z) = \overline{z} = x - iy.$$

Show that

(i) $T(z_1 + z_2) = T(z_1) + T(z_2)$, [2]

(ii) $\alpha T(z_1) = T(\alpha z_1)$, where $\alpha \in \mathbb{R}$, [2]

(iii) $\alpha T(z_1) \neq T(\alpha z_1)$, where $\alpha \in \mathbb{C}$, [4]

where $z_1, z_2 \in \mathbb{C}$ with $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

(b) Let $S : \mathbb{R}^2 \to \mathbb{R}$ be an operator given by

$$S\big[(x, y)\big] = \frac{x^2}{y}.$$

Show that

(i) $S\big[(x_1, y_1) + (x_2, y_2)\big] \neq S\big[(x_1, y_1)\big] + S\big[(x_2, y_2)\big]$, [4]

(ii) $S\big[\alpha(x_1, y_1)\big] = \alpha S\big[(x_1, y_1)\big]$, where $\alpha \in \mathbb{R}$. [4]

(c) State, giving reasons which of the above operators, T and S are linear over the field of **real numbers**. [4]

B8. Let X be a space and consider the metric function d given by

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i - y_i|}{1 + |x_i - y_i|}. \tag{2}$$

(a) Let a and b be any real or complex numbers. Prove that

$$\frac{|a + b|}{1 + |a + b|} \leq \frac{|a|}{1 + |a|} + \frac{|b|}{1 + |b|}.$$

[5]

(b) (i) Use part a) to prove the triangle inequality for the metric function d. [5]

(ii) Verify the remaining properties of a metric space to prove that (X, d) is metric space. [6]

(c) A metric is induced by a norm if

$$d(x, y) = d(x + z, y + z),$$

and

$$d(\alpha x, \alpha y) = |\alpha| d(x, y),$$

for all $x, y, z \in X$ and every scalar α.

Prove that the metric d given by (2) is **not** induced by a norm on X. [4]

B9. (a) Let X be an inner product space. Define

    (i) orthogonality of vectors in X.   [3]

    (ii) orthonormality of vectors in X.   [3]

(b) Consider the Hilbert space $L^2(-\pi, \pi)$ with the following basis $\mathcal{B} = \{a_0, a_1, a_2\}$ where

$$a_0 = \frac{1}{\sqrt{2\pi}},$$
$$a_1 = \frac{1}{\sqrt{\pi}} \cos(t),$$
$$a_2 = \frac{1}{\sqrt{\pi}} \cos(2t).$$

The inner product on $L^2(-\pi, \pi)$ is given by

$$\langle x, y \rangle = \int_{-\pi}^{\pi} x(t)y(t)\, \mathrm{d}x.$$

    (i) Show that **all** pairs of members of $\mathcal{B}$ are orthogonal in $L^2(-\pi, \pi)$,   [7]

    (ii) Show that **all** pairs of members of $\mathcal{B}$ are orthonormal in $L^2(-\pi, \pi)$.   [7]

HINT:

You may make use of the following **product-to-sum** trigonometric formula

$$\cos A \cos B = \frac{1}{2}\Big[\cos(A - B) + \cos(A + B)\Big].$$

END OF QUESTION PAPER