NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA4236

FACULTY OF APPLIED SCIENCE

DEPARTMENT OF APPLIED MATHEMATICS

SMA4236: CONTROL THEORY

MARCH 2025 EXAMINATION

Time : 3 hours

Candidates should attempt **ALL** questions from Section **A** [**40 marks**] and **ANY THREE** Questions in Section **B** [**60 marks**].

SECTION A: Answer ALL questions in this section [40]

A1. Define the following phrases:

(a) A system. [2]

(b) A control function. [2]

(c) A system transfer function. [2]

(d) An optimal control problem. [5]

A2. Derive the formula for the system transfer function of a system made up of components whose transfer functions are $A_1(s)$, $A_2(s)$ and $A_3(s)$, connected in a parallel formation. [3]

A3. (a) Find Laplace transforms of $f(t) = e^{-bt}\cos(\omega t)$ and $g(t) = e^{-bt}\sin(\omega t)$. [4]

(b) Use Laplace transforms to solve $y'' + 2y' + 5y = 5$ given $y(0) = y'(0) = 0$. [3]

(c) Solve $y'' - 2y' + y = e^{3t},\ y(0) = 1,\ y'(0) = 2$, by using Laplace transforms. [3]

A4. (a) Consider the n^{th}-order autonomous system

$$y^n + a_1 y^{n-1} + a_2 y^{n-2} + \ldots + a_n y = u.$$

Show that any state of such a system of differential equations can be represented as $\dot{x} = A\underline{x} + B\underline{u}$. [5]

(b) Find the general solution of this equation. [6]

(c) Suppose the forcing function is removed from $\dot{x} = A\underline{x} + B\underline{u}$, explain how the resulting system can be solved, and give the general form of the final solution. [5]

SECTION B: Answer ANY THREE questions in this section [60]

B5. (a) Show that the state equations for a system whose forcing function involves derivative terms, for example,

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1}\dot{y} + a_n y = b_0 u^{(n)} + b_1 u^{(n-1)} + \ldots + b_{n-1}\dot{u} + b_n u$$

are

$$\underline{\dot{x}} = A\underline{x} + B\underline{u},$$
$$y = C\underline{x} + D\underline{u}.$$

[8]

(b) Hence obtain the state equations of the system $\ddot{y} + 2\dot{y} - y = 2\ddot{u} - \dot{u} + 3u$. [6]

(c) Find e^{At} if

$$A = \begin{pmatrix} 0 & 2 \\ -1 & -3 \end{pmatrix}.$$

[6]

B6. (a) Using the matrix A given in the preceding question, Question **B5(c)** and $f(x) = x^2 - 2x + 1$, find $f(A)$. [5]

(b) Given the non-homogeneous linear differential equation $\underline{\dot{x}} = A\underline{x} + u(t)$, let $Py = x$, where P is the modal matrix. Show that the system reduces to $\dot{y} = Dy + h(t)$, where $h(t)$ is a system of uncoupled equations, and give a formula for $h(t)$. [6]

(c) Show that $\dot{y}_i(t) = \lambda_i(t) y_i(t) + h_i(t)$. [3]

(d) Solve this differential equation for $y_i(t)$. [6]

B7. (a) Explain the phrases below:

(i) A stable control system. [3]

(ii) A marginally stable control system. [3]

(b) Describe how you would use the Routh-Hurwitz criterion to establish the stability of the system whose characteristic polynomial is

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0, \quad a_0 \neq 0.$$

[8]

(c) Hence, determine the stability of the system with the characteristic equation $2s^4 + s^3 + 3s^2 + 5s + 10$. [6]

B8. (a) Let

$$J(u) = \int_{t_0}^{t_1} f(t, x(t), u(t))dt$$

be the objective functional for an optimal control problem with $x = x(u)$, as its state and $u(t)$ as the corresponding control.

Derive the adjoint equation, the transversality and the optimality conditions. [16]

(b) Hence, solve the minimization problem

$$\min_u \int_0^1 u(t)^2 dt,$$

subject to

$$x'(t) = x(t) + u(t), x(0) = 1, x(1) = \text{free}.$$

[4]

END OF QUESTION PAPER