

NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY
SMA4241

FACULTY OF APPLIED SCIENCE
DEPARTMENT OF APPLIED MATHEMATICS
SMA4241 FINANCIAL MATHEMATICS

MARCH 2025 EXAMINATION

Time : 3 hours

ADDITIONAL MATERIAL: Statistical Tables

SECTION A: Answer ALL questions in this section [40].

A1. Define the following terms:

- (a) Sigma algebra. [2]
(b) Filtration. [2]

A2. Find the number of bonds of type A and B to be bought if $D_A = 2$, $D_B = 3.4$, $P_A = 0.98$, $P_B = 1.02$ and you need a portfolio worth \$5,000 with duration 6. [6]

A3. Find the the principal to be deposited initially in an account attracting simple interest at a rate of 6% per annum if \$1000 is needed after three months(91 days) . [4]

A4. Suppose that you deposit \$1,200 at the end of each year for 40 years, subject to annual compounding at a constant rate of 5%. Find the balance after 40 years. [6]

- A5.** Suppose the gold price $S(t)$ is modelled by a geometric Brownian motion $S(t) = \exp(\mu t + \sigma B_t)$, where B_t is the standard Brownian motion, $\mu = 0.062$, $\sigma = 0.1$ and $S(2) = 122$. Given that Mr Msobho is willing to buy the gold at time $t = 10$.
- (a) Find the probability that the price of gold at $t = 10$ is greater than 160. [5]
- (b) Find the variance and expected price of gold in at time $t = 10$. [5,5]
- (c) Check if the price process of gold is a martingale. [5]

SECTION B: Answer THREE questions in this section [60].

- B6.** (a) Define the following terms
- (i) Annuity. [2]
- (ii) Delta of a derivative. [2]
- (iii) Probability measure. [4]
- (b) Consider a three step binomial tree model. The initial stock price $S_0 = \$100$ and the risk free interest rate is $r = 0.01$. Take $u = 0.2$ and $d = -0.1$, assuming that a dividend of \$10 dollars is paid at time step 1. Find the option price for an American put option with strike price \$100 which expires in 12 months. [12]
- B7.** (a) Given that you are a financial adviser and your client told you that the yield $y(7) = 0.90$. Given that the current yield is $y(0) = 0.5$. Discuss fully what advise would you give to your client who is willing to get risk free profit. [10]
- (b) Compute the value after 4 years of 1420 dollars invested in a 4 year bond with 36 dollar annual coupons and 120 dollars face value if the consecutive rates are 11%, 11.5%, 12% and 9%. [10]
- B8.** Consider a two-period market model consisting of savings account and one risky stock. The interest rate $r = \frac{1}{9}$ so that $A_t = (1 + r)^t = (\frac{10}{9})^t$. The price of stock at time 0: $S_0 = 1.5$, at time period 1: $S_1^u = 2.2$, $S_1^d = 0.75$ with probability 0.6 and 0.4 respectively. At time period 2 if the stock price went up at time period 1 the possible stock prices are given as: $S_2^{uu} = 3$, $S_2^{ud} = 1$ with probabilities 0.3 and 0.7 respectively. At time period 2 if the stock price went down at time period 1 the possible stock prices are given as: $S_2^{du} = 1.4$, $S_2^{dd} = 0.6$ with probabilities 0.4 and 0.6 respectively. Consider a digital call option with the following payoff function

$$X(S_2(\omega)) = \begin{cases} 3.50 & \text{if } 1 \leq S_2(\omega) < 3 \\ 5.50 & \text{if } S_2(\omega) \geq 3 \\ 0 & \text{otherwise} \end{cases}$$

Find the initial price of this contingent. [20]

- B9.** (a) Consider a stock price process S_t which pays continuous dividend at rate q generated by a geometric Brownian motion under \mathbb{P} ,

$$dS_t = (\mu - q)S_t dt + \sigma S_t dB_t$$

with B_t a \mathbb{P} -Brownian motion. Also, consider a cash bond with price $A_t = e^{rt}$, so that

$$dA_t = rA_t dt.$$

Given that the payoff function at expiration date is defined as:

$$f(S_T) = \begin{cases} 0 & \text{if } S_T < 1 \\ 2.5 & \text{if } 1 \leq S_T \leq 5 \\ 6 & \text{if } S_T > 5 \end{cases}$$

Using the risk neutral approach derive the option pricing formula clearly stating assumptions used. [15]

- (b) A listed company on the ZSE has the stock price six months from expiration of an option as \$95, risk free interest rate is 4% per annum and an exercise price of \$90. The volatility is 30% per annum. Calculate the price of the European call option using the Black-Scholes option pricing model. [5]

END OF QUESTION PAPER