NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA5111

FACULTY OF APPLIED SCIENCE

DEPARTMENT OF APPLIED MATHEMATICS

SMA5111: FUNCTIONAL ANALYSIS

DECEMBER 2024 EXAMINATION

Time : 3 hours

Candidates should attempt all questions from Section A [**40 MARKS**] and ANY THREE questions in Section B [**60 MARKS**].

SECTION A

A1. Let X be the normed space whose points are sequences of complex numbers $x = \{\xi_j\}$ with only finitely many non-zero terms and norm defined by $\|x\| = \sup_j |\xi_j|$. Let $T : X \to X$ be defined by

$$Tx = \left(\xi_1, \frac{1}{2}\xi_2, \frac{1}{3}\xi_3, \dots\right) = \left(\frac{1}{j}\xi_j\right)$$

Show that T is linear and bounded, but T^{-1} is unbounded. Does this contradict the Bounded Inverse Theorem? [8]

A2. Let $\Omega \subseteq \mathbb{R}^m$ be an open subset and let $(\Omega_n)\, n \in \mathbb{N}$ be an exhaustion of Ω by open sets with compact closure, that is, each $\Omega_n \subseteq \mathbb{R}^m$ is open, $\bar{\Omega}_n$ is compact and contained in $\Omega, \Omega_n \subseteq \Omega_{n+1}$ and $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n$. Define

$$d(f,g) = \sum_{n\in\mathbb{N}} 2^{-n} \frac{\|f-g\|_{C^0(\bar{\Omega}_n)}}{1+\|f-g\|_{C^0(\bar{\Omega}_n)}}$$

for every continuous, real-valued functions $f, g \in C^0(\Omega, \mathbb{R})$.

Prove that d defines a metric in $C^0(\Omega, \mathbb{R})$. [7]

A3. The space $\ell^p, p > 1$ consists of all sequences $\mathbf{x} = \{x_n\}, x_n \in \mathbb{R}$ with the norm defined by

$$\|\mathbf{x}\| = (\sum_{i=1}^{\infty} |x_i|^p)^{\frac{1}{p}}, \quad \sum_{i=1}^{\infty} |x_i|^p < \infty.$$

Show that ℓ^p is a Banach space. [8]

A4. (a) Let X be a real vector space and p a sub-linear functional defined on X, which has the following properties.

(i) Positive Homogeneity

$$p(\alpha x) = \alpha p(x) \text{ for all } a>0$$

(ii) Sub-Additivity

$$p(x + y) \leq p(x) + p(y) \qquad \text{for all } x, y \in X$$

Y denotes a subspace of X on which a functional ℓ is defined that is dominated by p:

$$\ell(y) \leq p(y) \quad \forall y \in Y$$

Prove the assertion below :
$\exists\ L : X \to \mathbb{R}$ that has the following properties

(i) L is linear
(ii) $L(y) = \ell(y) \quad \forall y \in Y$
(iii) $L(x) \leq p(x) \quad \forall x \in X$

[15]

Note : In other words prove that ℓ can be extended to all of X as a linear functional dominated by p

(b) If p is a sublinear functional on a real vector space X, show that there exists a linear functional $\tilde{f}$ on X such that $-p(-x) \leq \tilde{f}(x) \leq p(x)$. [2]

SECTION B

Answer any **THREE** questions from this section **[60 MARKS]**

B5. (a) Let (X, d) be a complete metric space. Suppose $T : X \to X$ satisfies

$$d(T(x), T(y) \leq \alpha d(x, y)$$

For every $x, y \in X$, where $0 \leq \alpha < 1$. Prove that T has a unique fixed point in X, i.e., there exists only one $x \in X$ such that $T(x) = x$

[9]

(b) Consider the Fredholm integral equation

$$x(s) = y(s) + \mu \int_a^b K(s, t)x(t)dt$$

Where $K(s, t)$ is continuous on $[a, b] \times [a, b], y \in C[a, b]$ and $|K(s, t)| \leq \lambda$ for all $(s, t) \in [a, b] \times [a, b]$
Show that this integral equation has a unique solution on $[a, b]$.

[6]

(c) Show that $T : \mathbb{R} \to \mathbb{R}$ defined by

$$T(x) = \frac{\pi}{2} + x - \tan^{-1} x$$

has no fixed point, and

$$|T(x) - T(y)| < |x - y| \quad \text{for all } x \neq y \in \mathbb{R}.$$

Why doesn't this example contradict the contraction mapping theorem? [5]

B6. (Heisenberg's Uncertainty Principle) Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space over $\mathbb{C}$. Let $D_A, D_B \subset H$ be dense subspaces and let $A : D_A \subset H \to H$ and $B : D_B \subset H \to H$ be symmetric linear operators. Assume that

$$A(D_A \cap D_B) \subset D_B \quad \text{and} \quad B(D_A \cap D_B) \subset D_A,$$

and define the commutator of A and B as

$$[A, B] : D_{[A,B]} \subset H \to H, \quad [A, B](x) \mapsto A(Bx) - B(Ax),$$

where $D_{[A,B]} := D_A \cap D_B$.

(a) Prove that

$$|\langle x, [A, B]x\rangle_H| \leq 2\|Ax\|_H \|Bx\|_H \quad \text{for every } x \in D[A, B].$$

[3]

(b) Define now the standard deviation of A

$$\varsigma(A, x) := \sqrt{\langle Ax, Ax\rangle_H - \langle x, Ax\rangle_H^2}$$

at each $x \in D_A$ with $\|x\|_H = 1$. Verify that $\varsigma(A, x)$ is well-defined for every x (i.e. that the radicand is real and non-negative) and prove that for every $x \in D[A, B]$ with $\|x\|_H = 1$ there holds

$$|\langle x, [A, B]x\rangle_H| \leq 2\varsigma(A, x)\varsigma(B, x).$$

Remark. The possible states of a quantum mechanical system are given by elements $x \in H$ with $\|x\|_H = 1$. Each observable is given by a symmetric linear operator $A : D_A \subset H \to H$. If the system is in state $x \in D_A$, we measure the observable A with uncertainty $\varsigma(A, x)$. [10]

(c) Consider the Hilbert space $(H, \langle\cdot,\cdot\rangle_H) = (L^2([0, 1], \mathbb{C}), \langle\cdot,\cdot\rangle_{L^2})$ and the subspace

$$C_0^1([0, 1], \mathbb{C}) := \left\{f \in C^1([0, 1], \mathbb{C}) \mid f(0) = 0 = f(1)\right\}.$$

Recall that $C_0^1([0, 1], \mathbb{C}) \subset L^2([0, 1], \mathbb{C})$ is a dense subspace. The operators

$$P : C_0^1([0, 1], \mathbb{C}) \to L^2([0, 1], \mathbb{C}), \quad Q : L^2([0, 1], \mathbb{C}) \to L^2([0, 1], \mathbb{C})$$
$$f(s) \mapsto if'(s) \qquad\qquad f(s) \mapsto sf(s)$$

correspond to the observables momentum and position.

Show that P and Q form a Heisenberg pair and conclude that the uncertainty principle holds: for every $f \in C_0^1([0, 1], \mathbb{C})$ with $\|f\|_{L^2([0,1],\mathbb{C})} = 1$ there holds

$$\varsigma(P, f)\varsigma(Q, f) \geq \frac{1}{2}$$

Thus we conclude: The more precisely the momentum of some particle is known, the less precisely its position can be known, and vice versa. [7]

B7. (a) Show that the space $(C[0,1], \|\cdot\|_\infty)$, where $\|x\|\infty = \sup_{0\leq t\leq 1}|x(t)|$, is not an inner product space, hence not a Hilbert space. [4]

(b) Let a, b be real numbers such that $a < b$. Consider the Hilbert space $L^2[a,b]$ over $\mathbb{R}$ and the operator $T : L^2[a,b] \to \mathbb{R}$ defined by

$$Tf = \int_a^b f(x)dx, \quad f \in L^2[a,b]$$

(i) Show that T is bounded. Compute $\|T\|$. [7]

(ii) According to the Riesz's Theorem, there exists a function $g \in L^2[a,b]$ such that

$$Tf = \langle f, g\rangle \text{ for all } f \in L^2[a,b]$$

Find such a function g and verify that $\|g\|_{L^2} = \|T\|$. [2]

(c) Let H be a Hilbert space.

(i) Prove that for any two subspaces M, N of H we have

$$(M+N)^\perp = M^\perp \cap N^\perp$$

[5]

(ii) Prove that for any two closed subspaces E, F of H we have

$$(E \cap F)^\perp = \overline{E^\perp + F^\perp}$$

[2]

B8. (a) Define a bounded linear operator [2]

(b) State the Uniform Boundedness Theorem and prove it [11]

(c) If X and Y are Banach spaces and $T_n \in B(X,Y), n = 1, 2, \cdots$, show that the following are equivalent :

(i) $(\|T_n\|)$ is bounded,

(ii) $(\|T_n x\|)$ is bounded for all $x \in X$,

(iii) $(|g(T_n x)|)$ is bounded for all $x \in X$ and all $g \in Y'$.

[7]

END OF QUESTION PAPER