NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

FACULTY OF APPLIED SCIENCE

DEPARTMENT OF APPLIED MATHEMATICS

SMA5112: VARIATIONAL CALCULUS

DECEMBER 2024 EXAMINATION

Time : 3 hours

Answer **ALL** Questions in Section **A** and **THREE** Questions in Section **B**.

SECTION A: Answer ALL questions in this section [40].

A1. Obtain the Euler's equation to the extremal of

$$\int_{x_1}^{x_2} \left(y(x)^2 - y(x)y'(x) + y'(x)^2\right) dx.$$

[4]

A2. Prove that the shortest distance between two points on a circular cylinder, when the points are not on a generator is along a circular helix joining them.
Hint: Let $x^2 + y^2 = a^2$ be the equation of the circular cylinder with the $z-$axis as its axis. Let $\rho,\ \phi,\ z$ be the cylindrical coordinates such that

$$ds = \sqrt{(d\rho)^2 + \rho^2(d\phi)^2 + (dz)^2}.$$

[7]

A3. Find the curve that minimizes the functional

$$I[y] = \int_a^b \sqrt{1+y'^2}dx, \quad y(a) = A, \quad y(b) = B.$$

[6]

A4. Find the Euler equations corresponding to the following functional

$$I[u] = \int\int_{\mathbb{R}} (x^2u_x^2 + y^2u_y^2)dxdy.$$

[5]

A5. Show that if the Lagrangian is independent of y, then the Euler equations are given by

$$\frac{\partial F}{\partial y'} = C$$

[5]

A6. Find the plane curve of fixed perimeter L which encloses maximum area. [8]

A7. Find the extremal of $I[y] = \int_0^S \left(\frac{(y')^2}{2} - y + \frac{x}{2}\right) dx$ with $y(0) = 0$. [5]

SECTION B: Answer any THREE questions in this section [60].

B8. (a) Consider a layer of a transparent medium with refractive index given by $n(x, y) = \frac{2}{y}$ that is located in the part of the plane given by $\frac{1}{2} \leq y \leq 2$.

(i) Use the Fermat's principle to show that the path of a light ray located within this medium is described by the differential equation

$$\frac{dy}{dx} = \frac{1}{ky}\sqrt{1-k^2y^2}, \text{ for } \frac{1}{2} \leq \text{y} \leq 2.$$

where k is a constant. [8]

(ii) Find the path of a light ray from source at $(-1,\ 1)$ to observer $(1,\ 1)$. [7]

(b) Find the Euler equation for the functional

$$I[x] = \int_{x_1}^{x_2} (x + y^2 + 3y')dx,$$

[5]

B9. (a) Minimize the functional

$$I[y] = \int_0^1 \frac{1}{2}(y')^2 dx + Ay(1), \quad y(0) = 0.$$

[6]

(b) Extremize the problem

$$I[y] = \int_0^1 (y' - \cos x)^2 \, dx, \quad y(0) = 1, \quad y(1) = 1.$$

[8]

(c) A particle moves on the surface $\phi(x, y, z) = 0$ from the point (x_1, y_1, z_1) to the point (x_2, y_2, z_2) in the time T. Show that if this particles moves in such a way that the integral of its kinetic energy over that time $\left(\frac{1}{2}\int_0^T x'^2 + y'^2 + z'^2 dt\right)$ is a minimum, its coordinates must satisfy the equations

$$\frac{x''}{\phi_x} = \frac{y''}{\phi_y} = \frac{z''}{\phi_z}.$$

[6]

B10. (a) Minimize the functional

$$I[y] = \int_0^{\frac{\pi}{4}} \left(y^2 - y'^2\right) dx, \quad y(0) = 1$$

and the right hand end point is along the cuirve $x = \frac{\pi}{4}$. [7]

(b) Consider the Dirichlet minimization problem

$$I(u) = \int\int_\Omega \frac{1}{2}\left(u_x^2 + u_y^2\right) dxdy.$$

Find the Euler -Lagrange equations. [6]

(c) Derive the Euler-Lagrange equations for the minimum surface problem

$$I(u) = \int\int_\Omega \left(\sqrt{(1 + u_x^2 + u_y^2)}\right) dxdy.$$

[7]

B11. (a) Consider a set of all closed non intersecting plane curves P, for which the total length is L. Find the curve that enclosed the greatest area. [7]

(b) Find the maximum area under the any curve of fixed length L going from $(x_0,\ 0)$ to $(x_1,\ 0)$ with $x_0 < x_1$ in the upper half plane. [13]

END OF QUESTION PAPER