NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA5161

FACULTY OF APPLIED SCIENCE

DEPARTMENT OF APPLIED MATHEMATICS

SMA5161: NUMERICAL SOLUTIONS FOR ODEs

DECEMBER 2024 EXAMINATION

Time : 3 hours

Candidates should attempt **ANY FOUR**. Each question carries **25** marks.

A1. (a) Define the Lipschitz condition. [5]

(b) Show that the initial-value problem below is well-posed on the domain $D = \{(t,y)\,|0 \leq t \leq 2\}$ and $-\infty < y < \infty$

$$\frac{dy}{dt} = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5$$

[8]

(c) Hence, use the Euler's method with step size of $h = 0.5$ to approximate the solution of the problem and compare the solution with the actual solution $y(t) = (t+1)^2 - 0.5e^t$, and also show the results in tabula form. [12]

A2. (a) State and explain any three possible sources of error in numerical analysis. [6]

(b) Consider the following linear system of equations

$$\begin{aligned} 4x - y + z &= 7 \\ -x + 3y - z &= 4 \\ x + y + 5z &= 6 \end{aligned}$$

Solve the system directly by Gaussian Elimination. [3]

(c) Suppose we intend to approximate the solution of the system with initial approximation $x_0 = y_0 = z_0 = 0$.

(i) Write the Jacobi iteration in the form

$$\underline{X}_{k+1} = T\underline{X}_k + \underline{c}.$$

[3]

(ii) Write the Gauss-Seidel iteration in the form

$$\underline{X}_{k+1} = \widetilde{T}\underline{X}_k + \widetilde{\underline{c}}.$$

[4]

(iii) Determine which algorithm converges (Perform four iterations). [9]

A3. (a) Determine by Taylor expansion the order of the following two approximations

1.
$$y'(x) \approx \frac{(-3y(x) + 4y(x+h) - y(x+2h))}{2h},$$

[4]

2.
$$y''(x) \approx \frac{(y(x+h) - 2y(x) + y(x-h))}{h^2}.$$

[4]

(b) Derive a second order difference approximation to $y'(x)$ using the values $y(x-2h)$, $y(x-h)$, $y(x+h)$ and $y(x+2h)$. [7]

(c) Consider the initial value problem

$$y'(x) = y - x^2, \quad y(0) = 2$$

on the interval $[0, 1]$ with $h = 0.1$.
Use the Adams-Bashforth-Moulton method to solve the differential equation at $x = 0.4$ to 8 significant figures. [10]

A4. (a) Define the following

(i) an implicit linear m-step multistep method. [4]

(ii) a stable linear m-step method. [3]

(b) Consider the linear multistep method

$$x_{j+3} - 2x_{j+2} + x_{j+1} = h(f_{j+3} + 2f_{j+2} + f_{j+1}).$$

(i) Determine whether the method is zero-stable. [6]

(ii) Show that the method is of order 3. [4]

(c) Consider the following system of first order differential equations;

$$\begin{aligned} \dot{y} &= -50y + z, \quad y(0) = 1; \\ \dot{z} &= -0.05z, \quad z(0) = 1. \end{aligned}$$

Determine the values of the step size h, for which the Euler forward method is stable. [8]

A5. As a mathematical modeler you have been given a task of fitting a simple Monkeypox model;

$$\begin{aligned} \dot{S} &= -\beta SI, \\ \dot{I} &= \beta SI, \end{aligned}$$

to the observed data given in the table below based on an outbreak in Democratic Republic of Congo 2022-2023.

Table 1: Daily Monkeypox Cases

Day	1	2	3	4	5
Cases	1	3	7	15	30

It is assumed that the initial susceptible individuals (S) and infected individuals (I) are 5000 and 1 respectively. The transmission coefficient is assumed to be equal to $\beta = 1.0 \times 10^{-4}$.

(a) Write a code in MATLAB that can be used to simulate the model using the in-built function ode45. [10]

(b) By using an appropriate goodness of fit metric and the forward Euler method check whether $\beta = 1.0 \times 10^{-4}$ is a better estimate compared to $\beta = 5.0 \times 10^{-4}$. [15]

END OF QUESTION PAPER