NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 5191

FACULTY OF APPLIED SCIENCE

DEPARTMENT OF APPLIED MATHEMATICS

SMA 5191: MATHEMATICAL MODELLING

DECEMBER 2024: EXAMINATION

Time : 3 hours

Candidates should attempt **ALL** questions from **Section A** (40 marks) and **ANY THREE** questions from **Section B** (20 marks each). GOOD LUCK!

SECTION A

A1. Describe the following concepts, giving their advantages, disadvantages and/or examples where necessary;

(a) Abstraction, [5]

(b) Idealization, [5]

(c) Dimensional analysis. [5]

A2. (a) Write down a continuous mathematical expression that represents that represents the situation where the temperature rises from 50 degrees at time zero to a smooth maximum 100 degrees after $50sec$ and decrease exponentially to 50 degrees asymptotically. [5]

(b) Instead of 50 degrees, assume that the initial temperature in (a) above is 10 degrees. Write a mathematical expression that represents this situation. [8]

A3. The equation used to describe heat transfer is given by

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + \frac{q}{\rho c_p},$$

with the associated boundary and initial conditions given as

$$-\kappa \frac{\partial T}{\partial x} = h(T - T_\infty) at x = 0,$$
$$T = T_\infty \ at \ x = L,$$
$$T = T_\infty \ at \ t = 0.$$

Use the following dimensionless parameters to nondimensionalize the system,

$$t = L^2/\alpha, \ x = L\bar{x}, \ T = T_\infty + (T_w - I_\infty)\theta.$$

[Hint: $\alpha = \kappa/\rho c_p$] [12]

SECTION B

B4. (a) (i) Describe the step when using the Buckingham Pi method [4]

(ii) Some children are playing with soap bubbles, and you become curious as to the relationship between soap bubble radius and the pressure inside the soap bubble. You reason that the pressure inside the soap bubble must be greater than atmospheric pressure, and that the shell of the soap bubble is under tension, much like the skin of a balloon. You also know that the property surface tension must be important in this problem. Not knowing any other physics, you decide to approach the problem using dimensional analysis. Establish a relationship between pressure difference $\Delta P = P_{inside} - P_{outside}$, soap bubble radius R, and the surface tension σ_s of the soap film. Dimensions are $[P] = [ML^{-1}T^{-2}]$, $[\sigma_s] = [MT^{-2}]$, $[R] = [L]$ [8]

(b) Assume that one decides to invest/save money at a bank that offers $r\%$ interest rate compounded monthly. Also assume that the bank charges a fixed monthly service fee for handling the account of $\$x$. Consider the case where one earns the interest first and then the bank charges are deducted after.

(i) Find an expression for the amount at each month. [4]

(ii) Determine the minimum that one has to deposit for the fund to grow in each case. [2]

(iii) The classical formula for compound interest is give by $A = P\left(1 + \frac{\widetilde{r}}{m}\right)^{mt}$. Here A is the accumulated amount, P is the principle (invested), $\widetilde{r}$ is the interest rate (p.a), m is the number of conversions (per year), and t is the time (in years). Compare this formula and the general formula you obtained in (a) above and explain any discrepancies. [2]

B5. (a) Ticket to a musical concert are sold at \$150. From the previous concert 15 000 tickets were sold at this price. We also know that if we increase the price of the ticket by \$1 we lose 6 customers. What is maximum price these tickets can be sold for in order to maximise profit?. [6]

(b) New information actually shows that if the price of the tickets go up by \$1, 10 customers are lost, \$2, 30 customers are lost, \$3, 60 customers and \$4, 100 customers are lost. Find the optimum price that will maximize revenue. [14]

B6. (a) A new fatal disease has hit a small island community. All the population of the island are susceptible to infection. Being infected by the disease reduces life expectancy of that person from 60 to 5 years. New people are added to the susceptible population at a constant rate of 1000 per year. It is estimated that each person interacts with 800 people each year on the island, but that the disease is only passed on 1 in 2500 of these interactions.

(i) Show that the disease can be modelled by

$$\frac{dS}{dt} = \alpha - \frac{\beta SI}{N} - \gamma S,$$

$$\frac{dI}{dt} = \frac{\beta SI}{N} - \delta I.$$

Explain all the variables and parameters in the model. Also state any other assumptions used. [3]

(ii) Evaluate all the equilibrium points in the model and determine their stability. [7]

(b) The model for the interaction of horses (x) and sheep (y) kept in a single plot is given by

$$\dot{x} = x\,(3 - 2x - 3y)\,,$$

$$\dot{y} = y\,(5 - 4x - 4y)\,.$$

(i) Find all equilibrium points and determine their stability. [8]

(ii) Plot the phase portrait for this model. [2]

B7. Those attending an event at the 2023 Olympics will have to first pass through security, through one of forty turnstiles at the entrance to the Olympic Park. It is estimated that it will take an average of 10 seconds for each person to pass through a turnstile. Only one person at a time can pass through a turnstile.

The spectators will then have to walk to the stadium, a distance of one kilometer, which will take an average of ten minutes. There are four entrances to the stadium itself, A, B, C and D, each with 10 turnstiles. Spectators can choose to pass through any of the turnstiles at any of the entrances. It will take an average of 12 seconds for a spectator to pass through a turnstile.

(a) Estimate the latest time that spectators should start arriving at the entrance of the Olympic Park in order that the crowd of four thousand spectators gain entrance to the stadium before the start of an event at 3pm. [10]

(b) Due to poor servicing of the turnstiles, normally half of the turnstiles will be down. Estimate the latest time that spectators should start arriving at the entrance of the Olympic Park in order that the crowd of four thousand spectators gain entrance to the stadium before the start of an event at 3pm. [10]

END OF QUESTION PAPER