NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

FACULTY OF APPLIED SCIENCE

DEPARTMENT OF APPLIED MATHEMATICS

SMA 5211:ADVANCED DYNAMICAL SYSTEMS

DECEMBER 2024 EXAMINATION

Time : 3 hours

Candidates should attempt **ALL** questions from **Section A** (40 marks) and **ANY THREE** questions from **Section B** (20 marks each). GOOD LUCK!

SECTION A

A1. Define the following terms:

(a) A dynamical system; [2]

(b) Chaos; [2]

(c) Bifurcation. [2]

A2. Show that the dynamical system given by $\dot{x} = r - x - e^{-x}$ undergoes a saddle node bifurcation. Note that the normal form for a saddle node bifurcation is given by $\dot{x} = r - x^2 or\ \dot{x} = r + x^2$. [10]

A3. Using the dynamical system $\dot{x} = -y + ax(x^2 + y^2)$, $\dot{y} = x + ay(x^2 + y^2)$, discuss whether linearisation is a reliable method. [10]

A4. (a) Distinguish between a flow and a map. [4]

(b) Using the difference equation $x_{n+1} = rx_n(1 - x_n^2)$, find the stationary points of the system and determine the range of values of r for which these are stable or unstable. [10]

SECTION B

B5. The Lorenz equation are given by

$$\dot{x} = \sigma(y - x), \dot{y} = rx - y - xz, \dot{z} = xy - z,$$

(a) Find all fixed points of the system. [4]

(b) Use linearisation to determine the stability of the origin. [6]

(c) Given the Liapunov function $V(x, y, z) = \frac{1}{\sigma}x^2 + y^2 + z^2$. Show that the origin is asymptotically stable for all trajectories whenever $r < 1$ [10]

B6. Given the map $x_{n+1} = rx_n(1 - x_n)$;

(a) Find all the stationary points. [4]

(b) Determine the parameter range for which they are stable or unstable. [6]

(c) Find the 2-cycle for this system, i.e, $x = f(f(x))$ and determine the range of values of r for which these are stable.
Hint: $\lambda = f'(p)f'(q)$, $x(rx - (r - 1))(r^2x^2 - r(1 + r)x + (r + 1))$. [10]

B7. (a) Consider an undamped harmonic oscillator, $\ddot{\theta} + \sin\theta = 0$. Find the potential for the system and plot the phase diagram for the system. [8]

(b) Now consider a damped harmonic oscillator $\ddot{\theta} + \beta\dot{\theta} + \sin\theta = 0$. Find the potential for the system and plot the phase diagram for the system. [8]

(c) Show that the system in (a) is conserved and the system in (b) is dissipative.
Hint: find $\frac{dE}{dt}$ [4]

B8. (a) Consider the system given by $\dot{r} = r(1 - r^2) + \mu r\cos\theta$, $\dot{\theta} = 1$. Show that there exists a limit cycle for this system. [10]

(b) Show that a dynamical system given by $\dot{x} = rlnx + x - 1$ undergoes a transcritical bifurcation.
Hint: the normal for of transcritical bifurcation is $\dot{x} = rx - x^2$ [10]

END OF QUESTION PAPER