



NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

FACULTY OF SCIENCE AND TECHNOLOGY EDUCATION

DEPARTMENT OF SCIENCE, MATHEMATICS AND TECHNOLOGY EDUCATION

MASTER OF SCIENCE EDUCATION DEGREE IN MATHEMATICS

Non-Linear Ordinary Differential Equations (PST6133)

Main Examination Paper

November 2024

This Examination Paper consists of 5 printed pages

Time Allowed:	3 hours
Total Marks:	100
Special Requirements:	None
Internal Examiner:	Mr Mkwelie N
External Examiner:	Dr Sunzuma G

INSTRUCTIONS

1. Candidates should answer any **FOUR** questions, being careful to number them A1 to A5.
2. Each question should start on a fresh page.
3. Each question carries 25 marks.
4. Omission of essential working will result in loss of marks.
5. Candidates will be penalised for use of wrong mathematical notation.

MARK ALLOCATION

QUESTION	MARKS
A1	25
A2	25
A3	25
A4	25
A5	25
TOTAL	100

A1.(a) (i) Find the fixed points of the following equations.

(i) $\dot{x} = x + 1$ [2]

(ii) $\dot{x} = x^3 - x$. [3]

ii) Construct the phase portraits for each equation in(i). [5]

(iii) Hence determine the nature (attractor, repeller or shunt) of each fixed point in (i). [4]

(b) The equation of motion of a spring–mass system with damping is

$$m \frac{d^2u}{dt^2} + c \frac{du}{dt} + ku = 0$$

where $m, c,$ and k are positive.

(i) Write this second order equation as a system of two first order equations using the standard substitution $x = u, y = \frac{du}{dt}$. [4]

(ii) Show that $x = 0, y = 0$ is a critical point, and analyze the nature and stability of the critical point as a function of the parameters $m, c,$ and k . [5]

A2. (a) Find the general solution of the linear system

$$\mathbf{X}' = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \mathbf{X}. \quad [11]$$

(b) Let $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -2 & 4 \end{pmatrix}$.

Find

(i) the Jordan form of matrix \mathbf{A} . [3]

(ii) the matrix \mathbf{M} which satisfies $\mathbf{J} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}$. [3]

(c) Find the matrix representation of the linear system

$$\begin{aligned} \dot{x}_1 &= x_1 + 2x_2 \\ \dot{x}_2 &= 2x_2 \end{aligned}$$

under the change of variables

$$\begin{aligned}x_1 &= y_1 + 2y_2 \\x_2 &= y_2\end{aligned}\tag{6}$$

A3. (a) Consider the one-dimensional equation

$$x' = x^2 + 3ax + 2a^2$$

where a is a real constant.

(i) Find the equilibrium points of the equation. [3]

(ii) The equation has qualitative different phase portraits depending on a . sketch the phase portraits for the equation when $a < 0$, when $a = 0$, and when $a > 0$. [9]

(b) Solve the system $\mathbf{X}' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 12t - 11 \\ -3 \end{pmatrix}$ on the interval $(-\infty, \infty)$.

using method of undetermined coefficients. [13]

A4. (a) Consider Predator-prey Model described by the non-linear system

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= x^2 - y - 1\end{aligned}$$

(i) Find all the fixed points of the system. [3]

(ii) For each of the fixed point, determine the linearisation of the system at each fixed point using the Jacobian matrix. [8]

(iii) Describe the type of fixed points you found in (i). [4]

(b) Determine the critical point (x_0, y_0) and then classify its type and examine its stability by making the transformations.

$$x = x_0 + u$$

$$y = y_0 + v$$

for the following equation;

$$\frac{dx}{dt} = x + y - 1, \quad \frac{dy}{dt} = 2x - y - 5.\tag{10}$$

A5.(a) Consider the system

$$\mathbf{X}' = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \mathbf{X}$$

- (i) Find the general solution of the system. [5]
- (ii) Hence sketch the phase portrait for the system. [5]

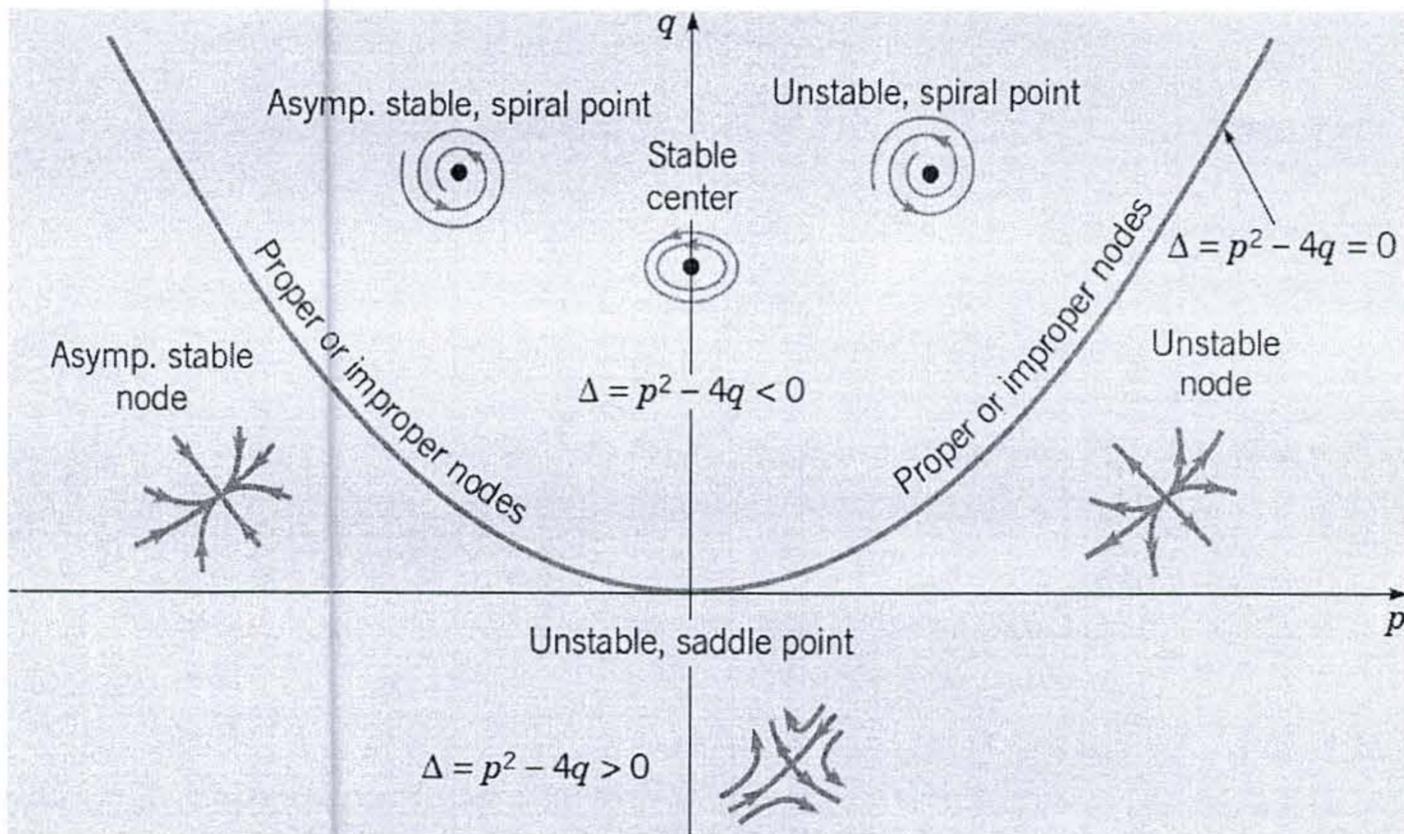
(b) Consider the autonomous system

$$\frac{dx}{dt} = f(x), \quad f(x) = -r \left(1 - \frac{x}{T}\right) \left(1 - \frac{x}{K}\right) x, \quad \text{where } r > 0 \text{ and } 0 < T < K.$$

- (i) Sketch the graph of $f(x)$ versus x . [3]
- (ii) Obtain the critical points and determine the nature of their stability and construct a phase portrait for each point. [9]
- (iii) Sketch the qualitative graph of $x(t)$ versus t [3]

END OF EXAMINATION PAPER

STABILITY DIAGRAM



STABILITY PROPERTIES OF LINEAR SYSTEMS

Eigenvalues	Type of Critical Point	Stability
$r_1 > r_2 > 0$	Node	Unstable
$r_1 < r_2 < 0$	Node	Asymptotically stable
$r_2 < 0 < r_1$	Saddle point	Unstable
$r_1 = r_2 > 0$	Proper or improper node	Unstable
$r_1 = r_2 < 0$	Proper or improper node	Asymptotically stable
$r_1, r_2 = \lambda \pm i\mu$	Spiral point	
$\lambda > 0$		Unstable
$\lambda < 0$		Asymptotically stable
$r_1 = i\mu, r_2 = -i\mu$	Center	Stable