NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

FACULTY OF SCIENCE AND TECHNOLOGY EDUCATION

DEPARTMENT OF SCIENCE, MATHEMATICS AND TECHNOLOGY EDUCATION

MASTER OF SCIENCE EDUCATION DEGREE IN MATHEMATICS

FUNCTIONAL ANALYSIS (PST 6332)

Main Examination Paper **November 2024**

This Examination Paper consists of 3 printed pages

Time Allowed: 3 hours

Total Marks: 100

Special Requirements: None

Internal Examiner: Mr Mkwelie N

External Examiner: Dr Sunzuma G

INSTRUCTIONS

1. Candidates should answer any **FOUR** questions, being careful to number them A1 to A6.
2. Each question should start on a fresh page.
3. Each question carries 25 marks.
4. Omission of essential working will result in loss of marks.
5. Candidates will be penalised for use of wrong mathematical notation.

MARK ALLOCATION

QUESTION	MARKS
A1	25
A2	25
A3	25
A4	25
A5	25
A6	25
TOTAL	100

A1.(a) (i)Define a metric on X. [3]

(ii) Consider the metric integral defined on $C[a,b]$ by

$$d(x,y) = \int_a^b |x(t) - y(t)|dt$$

Show that d is a metric. [8]

(b) Let (X,d) be a Metric Space and the metric induced by the norm on X is defined by $d(x,y) = \|x - y\|$.

(i) Prove that d defines a metric on X. [8]

(ii) Show that d satisfies the translation invariant properties,

$d(x + a, y + a) = d(x.y)$ and $d(\alpha x, \alpha y) = |\alpha| d(x,y)$. [6]

A2.(a) Let X be a Normed Space given by $\|x\|_p = \left[\sum_{i=1}^{\infty} |\xi_i|^p\right]^{1/p}$. Verify that $\|.\|_p$ defines a norm in l^p. [8]

(b). Suppose $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$, given that $f(x) = x + 2$ and $g(x) = 3x - 2$.

Evaluate

(i) $\langle f, g \rangle$ [4]

(ii) $\|f - g\|$ [4]

(c) Let (x_n) and (y_n) be sequences in an metric space (X,d), show that if

$x_n \to x$ and $y_n \to y$ in X then $d(x_n, y_n) = d(x,y)$ [9]

A3. (a) Let $(X, \langle .,. \rangle)$ be an Innner Product Space. Show that

(i) $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$ [3]

(ii) $\langle x, \lambda y \rangle = \bar{\lambda} \langle x, y \rangle$ [3]

(b) Prove that $(X, \langle . , . \rangle)$ is an Inner Product Space, where $x = \mathbb{R}^n$ and $\langle . , . \rangle$ is defined by $\langle x, y \rangle = \sum_{i=1}^{n} \xi_i \bar{\eta}_i$ such that $x = (\xi_1, \xi_2, \xi_3, \dots, \xi_n)$ and

$x = (\eta_1, \eta_2, \eta_3, \dots, \eta_n)$. [10]

(c) The functional space $C[0,1]$ with the norm defined by $\|x\|_\infty = \max_{0 \le t \le 1} |x(t)|$.

By taking $x(t) = 1$ and $y(t) = t$ Prove that the space $C[0,1]$ is not an Inner Product Space. [9]

A4. (a) Make use of the triangle inequality to show that

$|d(x,z) - d(y,z)| \le d(x,y)$. [5]

b).If x and y are orthogonal vectors Inner Product Space X. Prove that,

(i) $\|x + y\|^2 = \|x\|^2 + \|y\|^2$. [4]

(ii) $\|x - y\|^2 = \|x\|^2 + \|y\|^2$. [4]

(iii).Hence , show that $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$. [2]

(c) Use the Gram-Schmidt algorithm to find an orthogonal and othonormal basis for the subspace V of $\mathbb{R}^4$ spanned by

$u = (1, 1, 1, 1)$, $v = (1, 2, 4, 5)$ and $w = (1, -3, -4, -2)$. [10]

A5. (a) Define a linear operator. [3]

(b) Let X and Y be a normed linear space over F. Given that $B(X, Y)$ is a set of bounded linear operators with norm T defined by $\|T\|$. Prove that $B(X, Y)$ is a normed linear space over F. [7]

(c) Prove the Cauchy Schwarz inequality $|\langle x, y \rangle| \le \|x\|.\|y\|$, for all $x, y \in X$, [7]

(d) An Inner Product Space is a metric space defined by

$d(x, y) = \|x - y\| = \sqrt{(x - y, x - y)}$, $x, y \in$ X. Show that d satisfies all the properties of a metric on X. [8]

END OF EXAMINATION PAPER